

The 3/5-conjecture for weakly $S(K_{1,3})$ -free forests

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Abstract

The 3/5-conjecture for the domination game states that the game domination numbers of an isolate-free graph G on n vertices are bounded as follows: $\gamma_g(G) \leq \frac{3n}{5}$ and $\gamma'_g(G) \leq \frac{3n+2}{5}$. Recent progress have been done on the subject and the conjecture is now proved for graphs with minimum degree at least 2. One powerful tool, introduced by Bujtás is the so-called greedy strategy for Dominator. In particular, using this strategy, she has proved the conjecture for isolate-free forests without leafs at distance 4. In this paper, we improve this strategy to extend the result to the larger class of weakly $S(K_{1,3})$ -free forests, where a weakly $S(K_{1,3})$ -free forest F is an isolate-free forest without induced $S(K_{1,3})$, whose leafs are leafs of F as well.

Keywords: domination game; 3/5-conjecture

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1 Introduction

The domination game, introduced five years ago in [3] is played on an arbitrary graph G by two players, *Dominator* and *Staller*. They alternately choose a vertex from G such that at least one previously undominated vertex becomes dominated. The game ends when no move is possible. Dominator aims to end the game as soon as possible, while Staller wants to prolong it. By *D-game* (resp. *S-game*) we mean a game in which Dominator (resp. Staller) plays first. Assuming that both players play optimally, the *D-game domination number* $\gamma_g(G)$ (resp. the *S-game domination number* $\gamma'_g(G)$) of a graph G , denotes the total number of chosen vertices during D-game (resp. S-game) on G . There is already a flourishing amount of works on the subject, see for various examples [15, 10, 1, 5, 2, 16]. The most outstanding conjecture concerning this game is probably the so-called 3/5-conjecture. (For related developments concerning the total domination game see [12, 13].)

Conjecture 1.1 (*Kinnersley, West and Zamani, [14]*) *If G is an isolate-free graph on n vertices, then*

$$\gamma_g(G) \leq \frac{3n}{5} \text{ and } \gamma'_g(G) \leq \frac{3n+2}{5}.$$

If true, the bounds are known to be tight, even if we restrict ourself to forests. See [4] for a study of the structure of graphs reaching these bounds. Recently Bujtás has made a breakthrough by introducing a powerful greedy strategy for Dominator [7]. In particular, using this technique, she has proved the conjecture for a large subclass of forests [7, 9]. In that case, the upper bound for S-game turns two be even better.

Theorem 1.2 (**Bujtás 2014**) *If F is an isolate-free forest on n vertices, without leafs at distance 4, then*

$$\gamma_g(F) \leq \frac{3n}{5} \text{ and } \gamma'_g(F) \leq \frac{3n+1}{5}.$$

This greedy strategy could also be applied to graphs with minimum degree at least 3. In [8], the conjecture is proved for this class of graph. Even more recently, Henning and Kinnersley [11] established the truth of the 3/5-conjecture over the class of graphs with minimum degree at least 2. Hence the 3/5-conjecture remains open only for graphs with pendant vertices. In this paper, we prove the 3/5 conjecture for a larger class of forests, which contains the forests without leafs at distance 4. Even if forests deserve interest by themselves, we emphasize that the general conjecture cannot be easily reduced to the one on forests. There are actually graphs which have greater domination number than any of their spanning trees [6].

Let $S(K_{1,3})$ be the graph obtained by subdividing once all the edges of the star $K_{1,3}$ (see Figure 1). A *weakly $S(K_{1,3})$ -free forest* is a forest which does not contain:

- isolated vertices and
- induced $S(K_{1,3})$ whose leafs are leafs in the original forest.

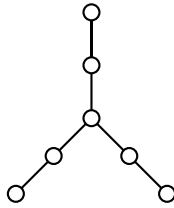


Figure 1: $S(K_{1,3})$

The class of weakly $S(K_{1,3})$ -free forests obviously contains the isolate-free forests without leafs at distances 4. For this class of forests we are able to extend Bujtás greedy strategy to prove the following more general theorem.

Theorem 1.3 *If F is a weakly $S(K_{1,3})$ -free forest on n vertices, then*

$$\gamma_g(F) \leq \frac{3n}{5} \text{ and } \gamma'_g(F) \leq \frac{3n+2}{5}.$$

The upper bounds are tight in both cases. For example, by [4, Theorem 3.7], trees from Figure 2 (a) reach the bound for D-game. For S-game, we easily verify that forests from Figure 2 (b) attain the bound as well.

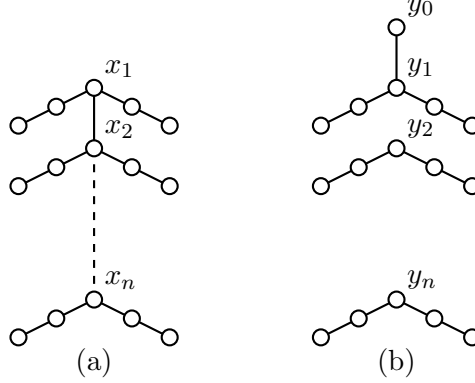


Figure 2: Extremal weakly $S(K_{1,3})$ -free forests

In the second section of the article, we are going to prove the upper bounds of our main theorem. The first subsection describes Bujtás' greedy strategy and introduces an improvement of it, the so-called breaking- P_5 strategy. The second subsection deals with the proof of the main theorem for D-game. Finally, in the last subsection, we explain briefly how to modify the proof to extend the result to S-game.

2 Proof of the main theorem

We start by proving the upper bound for D-game. All the results of the two following subsections are stated assuming that D-game is played. In the last subsection, we explain how to modify the proof to get the desired upper bound for S-game.

2.1 The greedy strategy and the breaking- P_5 strategy

First we recall Bujtás' greedy strategy for Dominator and give definitions and background needed to understand it. Our strategy will be compatible with the greedy one. In fact, it is only a refinement of the first phase of this strategy. Let F be an isolate-free forest on n vertices. All vertices will be colored with one of the following three colors: white (W), blue (B) and red (R). In addition to colors, each vertex of F has also a value. It is this value which enables to define a greedy like strategy. The coloring and the values will change all along the game. At the end of turn $k \geq 0$, the color and value of a vertex are defined as follows.

- A vertex is white and its value is 3 if it is not already dominated.
- A vertex is blue and its value is 2 if it is dominated, but at least one of its neighbors is not.
- A vertex is red and its value is 0 if $N[v]$ is entirely dominated.

Note that, when a vertex is selected by a player, it is always going to turn red during the current turn. We denote respectively by $W_k(F)$, $B_k(F)$ and $R_k(F)$, the set of white, blue and red vertices at the end of turn $k \geq 0$. When F will be clear from the context, we will omit it in the notations. At the end of turn $k \geq 0$, we define the residual forest F_k as the forest obtained from F by deleting all the red vertices and all the edges between two blue vertices (note that $F_0 = F$). Since all the legal moves are either white or blue vertices and all the blue vertices are already dominated, the game after k moves is the same played on the forest F or played on the residual forest F_k . The following statement proved in [9] is particularly useful. In particular, the first item ensures that any white leaf in a residual forest is also a leaf in the original forest. In other words, no white leaf is created during the game.

Lemma 2.1 *Let F be an isolate-free forest and $k \geq 0$.*

1. *If v belongs to W_k , then v has the same neighborhood in F_k as in F .*
2. *If F has no isolated vertex, then neither has F_k .*

For $k \geq 0$, the value of the residual forest F_k is the sum of the values of all its vertices. We denote this value by $p(F_k)$. By definition, $p(F) = 3n$. At each turn, the value of the residual forest decreases. We say that a player seizes s points during turn $k \geq 1$, when $p(F_{k-1}) - p(F_k) = s$. In the greedy strategy the game will be divided into four phases described below. Dominator will always start by applying, if possible, Phase 1 of the strategy. For $i \in \{2, 3, 4\}$, Phase i starts at Dominator's turn only if Phase $i - 1$ is no longer possible. If at this moment, the strategy of Phase i is not applicable, then it will be skipped. When Dominator starts to play in a new phase, he will never go back to a previous one, even if a change in the residual graph might cause this phase applicable again.

Greedy strategy.

- **Phase 1.** At his turn Dominator seizes at least 7 points and at least two vertices turn red during this turn.
- **Phase 2.** At his turn Dominator seizes at least 7 points.
- **Phase 3.** At his turn Dominator seizes at least 6 points. Moreover, he applies the following two rules.
 - Dominator selects a vertex which ensures the maximal possible gain achievable at this turn.

- Under the above rule, Dominator always prefers to play a white stem in the residual forest, which has a white leaf neighbor.

- **Phase 4.** At his turn Dominator seizes at least 3 points.

Roughly speaking, Bujtás' proof is to show that during a turn of Dominator and the turn of Staller which follows, the value of the residual forest is in average decreased by 10 points. It implies directly that after at most $3n/5$ moves, the value of the residual graph is 0, which means the game is over. But it works only if we do not deal with what Bujtás called critical P_5 .

Definition 2.2 *A critical P_5 in F_k , with $k \geq 0$ is a path of length 4 such that the following holds.*

- *All the vertices are in W_k , save the center of the path which is in B_k .*
- *Both ends of the path are leafs in F_k .*

The blue center of a critical P_5 is called a *critical center*. It is proved in [9], that these critical centers could only appear during Phase 1. Since no new white leaf is created along the game, the following observations are also easy to prove.

Observation 2.3 *If a vertex in B_k is not a critical center at the end of turn $k \geq 1$, then it is not a critical center at the end of any turn $k' \geq k$.*

In order to state precisely the results obtained by Bujtás, we need more definitions related to this phase. Let $k^* \geq 0$ be the turn at the end of which Phase 1 is over. If Phase 1 is skipped, it is easy to show that the 3/5-conjecture holds (the forest is actually a disjoint union of edges). For, we assume in the rest of the paper that $k^* \geq 1$. The number of critical centers at the end of turn k^* will be denoted by c^* . For each turn of Phase 1, when Dominator decreases the value of the graph by more than 7, he does better than what it is expected during this phase. It is the same for Staller when she decreases the value by more than 3 points. These bonus points are defined as follows.

Definition 2.4 *For $k \in \{1, \dots, k^*\}$, set:*

$$e_k = \begin{cases} p(F_{k-1}) - p(F_k) - 7 & \text{if turn } k \text{ is due to Dominator,} \\ p(F_{k-1}) - p(F_k) - 3 & \text{if it is due to Staller.} \end{cases}$$

The sum of all the bonus points earned during Phase 1 will be denoted by e^* , that is $e^* = e_1 + \dots + e_{k^*}$. In [9, Theorem 2], Bujtás proved the following result.

Proposition 2.5 *Let F be an isolate-free forest on n vertices. If Dominator plays D-game according to the greedy strategy, then the number of turns will be at most*

$$\frac{3n - e^* + c^*}{5}.$$

By Lemma 2.1, no new white leaf is created all along the game. It follows that for an isolate-free forest F without leafs at distance 4, c^* is always equal to 0. Hence, the above proposition shows that the 3/5-conjecture holds for D-game when restricted to this class of forests. For S-game similar arguments show that for this restricted class the upper bound is slightly better than the one announced by the conjecture: $\gamma'_g(F) \leq \frac{3|V(F)|+1}{2}$.

We now improve the greedy strategy in order to apply it to the larger class of weakly $S(K_{1,3})$ -free forests. Let F be an isolate-free forest on n vertices. From now on, we assume that F is arbitrarily rooted. More precisely, if F is a forest with $m \geq 1$ connected components T_1, \dots, T_m , then for any $i \in \{1, \dots, m\}$ we select some $r_i \in V(T_i)$ to be the root of the tree T_i . For every vertex $v \in V(T_i)$, the *height* of v is defined by $h(v) = d(r_i, v)$.

We define $\mathcal{P}(F)$ as the set of paths of length 4 between two leafs of F . We say that a vertex $d \in V(F)$ is a *dangerous center after $k \geq 0$ moves*, if the followings holds.

- The vertex d is the center of a path P in $\mathcal{P}(F)$.
- All the vertices of P are in W_k .

The set of dangerous centers after $k \geq 0$ moves will be denoted by $D_k(F)$. When F will be clear from the context, we will only write D_k . Because no new white leaf is created in the residual forests along the game (see Lemma 2.1), the two following results clearly hold.

Lemma 2.6 *If $k' \geq k \geq 0$ then $D_{k'} \subseteq D_k$, that is no new dangerous center is created during the game.*

Lemma 2.7 *If a vertex in W_k does not belong to D_k , with $k \geq 0$, then it could not be a critical center at the end of any turn $k' > k$.*

The following lemma is important, because it gives an alternative definition of the class of weakly $S(K_{1,3})$ -free forests.

Lemma 2.8 *If F is a weakly $S(K_{1,3})$ -free forest and d is a dangerous center in $D_0(F)$, then there are exactly two vertices $a_1, a_2 \in N(d)$, such that any path in $\mathcal{P}(F)$ whose center is d contains a_1 and a_2 .*

Proof. Let $l_1 a_1 d a_2 l_2$ be a path in $\mathcal{P}(F)$ whose center is the dangerous center d . Let f be a leaf of a path P in $\mathcal{P}(F)$ whose center is also d . We have to prove that the unique neighbor u of f is either a_1 or a_2 . By way of contradiction, assume that P contains neither a_1 nor a_2 . Then the subtree induced by $\{l_1, l_2, f, a_1, a_2, u\}$ is clearly isomorphic to $S(K_{1,3})$. Since l_1, l_2 and f are leafs of F , the forest is not weakly $S(K_{1,3})$ -free. A contradiction. \square

At least one of the two vertices a_1, a_2 defined in Lemma 2.8 has a height strictly greater than d . Without loss of generality, we always assume that $h(a_2) > h(d)$ and

we say that a_2 is the P_5 -child of d . We introduce now the breaking- P_5 strategy. The notation a_1 and a_2 will always refer to the two vertices defined in the above lemma. We will say that a_1 is the vertex *related* to the dangerous center d .

Breaking- P_5 strategy.

- **Phase 1.0:** if D_{k-1} is not empty at his turn $k \geq 1$, Dominator selects a vertex $d \in D_{k-1}$ with maximum height and he plays the vertex a_1 related to d . To emphasize that d is not the vertex played during this turn, we will always say that d is *elected*.
- **Phase 1.1:** at his turn, Dominator gets at least 7 points and at least two vertices turn red.
- **Phases 2 to 4:** these phases are the same as for the greedy strategy.

The phases follow one another in the same way as for the greedy strategy. We write $k^{**} \geq 0$ for the number of turns in Phase 1.1, c^{**} for the number of critical centers at the end of turn k^{**} and $e^{**} = e_1 + \dots + e_{k^{**}}$. If $k^{**} = 0$, then we are in the settings of Bujtás' proof. Hence, we assume now that $k^{**} \geq 1$. As we will see in the following proposition, the breaking- P_5 strategy is also a greedy strategy and we have a result similar to Proposition 2.5.

Proposition 2.9 *Let F be an isolate-free forest on n vertices. If Dominator plays D-game according to the breaking- P_5 strategy, then the number of turns will be at most $\frac{3n - e^{**} + c^{**}}{5}$.*

Proof. Let d be the dangerous center elected by Dominator according to the rules of Phase 1.0. Let a_1 be the neighbor of d defined in Lemma 2.8. Since Dominator plays according to the breaking- P_5 strategy, he plays a_1 . This vertex and at least one of its white leaf neighbors turn from white to red. Moreover, the elected vertex d turns from white to blue. Hence Dominator sizes at least 7 points, which in turn implies that the breaking- P_5 strategy is a greedy strategy. Applying Proposition 2.5, we have that D-game will end in at most $\frac{3n - e^* + c^*}{5}$ turns. Moreover, any critical center created at turn $k > k^{**}$ must belong to D_{k-1} . But $D_k = \emptyset$, for all $k \geq k^{**}$. Hence, by Lemma 2.7, we have $c^* = c^{**}$. Finally, it is obvious that $e^* \geq e^{**}$. In conclusion, if Dominator follows the breaking- P_5 strategy, D-game ends in at most $\frac{3n - e^{**} + c^{**}}{5}$ turns. \square

2.2 The proof for D-game

We now prove that if F is a weakly $S(K_{1,3})$ -free forest on n vertices, then $\gamma_g(F) \leq \frac{3n}{5}$. For this purpose, we introduce two new processes for each turn of the game. These processes come in addition to the coloring and values defined in the greedy strategy and they only concern blue vertices. When turning blue, some vertices will also be

highlighted. We denote by H_k the set of highlighted vertices at the end of turn $k \geq 1$. The highlighting process will only occur at Dominator's turns. Moreover all blue vertices will get either weight 0 or 1. We denote by $w_k : B_k \rightarrow \{0, 1\}$ the weight function at the end of turn $k \geq 1$. For $v \in V(F)$ and $X, Y \in \{B, R, W\}$, we write $v : X \rightsquigarrow Y$ to indicate that the color of v changes from X to Y during the current turn.

Highlighting Process.

- When Dominator elects the dangerous center $d \in D_{k-1}$ at turn $k \geq 1$, d is highlighted.

Note that $H_k \subseteq B_k$, because the elected vertex always turns blue during the turn where it is elected.

Weighting Process.

- Dominator's turn k , $k \geq 1$. All new blue vertices which are not highlighted get weight 1. The new highlighted vertex d gets weight 0.
- Staller's turn k , $k \geq 1$. She selects vertex v .
 - (A) $v : W \rightsquigarrow R$ and v is not the P_5 -child of a vertex $d \in H_{k-1}$. All new blue vertices get weight 1.
 - (B) $v : W \rightsquigarrow R$ and v is the P_5 -child of a vertex $d \in H_{k-1}$. All new blue vertices get weight 0 and the vertex d gets weight 1.
 - (C) $v : B \rightsquigarrow R$, $v \notin H_{k-1}$ and v is not the P_5 -child of a vertex $d \in H_{k-1}$. All new blue vertices get a weight equal to the weight of v .
 - (D) $v : B \rightsquigarrow R$, $v \notin H_{k-1}$ and v is the P_5 -child of a vertex $d \in H_{k-1}$. All new blue vertices get weight 0 and d get weight 1.
 - (E) $v : B \rightsquigarrow R$ and $v \in H_{k-1}$. In particular v is in D_0 . Let a_2 be the P_5 -child of v .
 - (a) If $a_2 \in W_{k-1}$, all new blue vertices but a_2 get weight 1 and a_2 gets weight 0.
 - (b) If $a_2 \in B_{k-1}$, all new blue vertices get weight 1 and a_2 gets new weight 0.
 - (c) If $a_2 \in R_{k-1}$, all new blue vertices get weight 1.
- Additional process for turn k , $k \geq 1$. This process is applied after the two former processes. If the P_5 -child of a vertex $d \in H_{k-1}$ turns from blue to red without being played by one of the player, then d gets a weight of 1.

Before being able to complete the proof, we need some technical definitions and lemmas. Until the end of this subsection, we always assume that Dominator plays according to the breaking- P_5 strategy.

Definition 2.10 *Let u be a vertex of a rooted forest F . For each turn $k \geq 0$, we define the following connected subtrees of $F \setminus R_k$.*

- $C_k(u)$ is the largest connected subtree of $F \setminus R_k$ which contains u .
- $C_k^+(u)$ is the largest connected subtree of $F \setminus R_k$ which contains u and whose vertices have height greater or equal to $h(u)$.
- $C_k^-(u) = C_k(u) \setminus C_k^+(u)$.

Remark that all these three sequences are non increasing with respect to inclusion. The following result follows directly from the definition.

Lemma 2.11 *Let F be a rooted forest. For all $k \geq 0$ and all $u, v \in V(F)$, if v is in $C_k^+(u)$ (resp. in $C_k^-(u)$), then $C_k^+(v) \subseteq C_k^+(u)$ (resp. $C_k^-(v) \subseteq C_k^-(u)$).*

Lemma 2.12 *Let F be a rooted forest and d be a vertex in $D_0(F)$, which is highlighted at turn $k \geq 1$. Then $C_{k'}^+(d) \cap D_{k'} = \emptyset$, for all $k' \geq k$.*

Proof. According to the prescribed strategy for Dominator, a dangerous center is highlighted at turn $k \geq 1$ only if it has a maximum height among all the dangerous center from D_{k-1} . In other words, $C_{k-1}^+(d) \cap D_{k-1} = \{d\}$. Moreover, $C_{k'}^+(d) \subseteq C_{k-1}^+(d)$ and $D_{k'} \subseteq D_{k-1}$, for all $k' \geq k$. Hence, $C_{k'}^+(d) \cap D_{k'} \subseteq \{d\}$. Finally, since d belongs to B_k , it is not in $D_{k'}$. We conclude that $C_{k'}^+(d) \cap D_{k'} = \emptyset$. \square

Lemma 2.13 *Let F be a rooted forest. Let d and d' be two vertices in $D_0(F)$ which are respectively highlighted at turns k and $k' \geq 1$. Then the following hold.*

- (i) *The P_5 -child a_2 of the vertex d belongs to W_k . Moreover, if a_2 is going to turn red or blue during Phase 1.0, this could be only during a turn of Staller.*
- (ii) *The vertex d is not the P_5 -child of d' .*

Proof. We prove (i) first. Let d be a dangerous center highlighted at turn k . The vertex d has to belong to D_{k-1} . By definition of the sets of dangerous centers, this implies that its P_5 -child a_2 is in W_{k-1} . Since d is highlighted at turn k this turn belongs to Dominator and during this turn he plays the vertex a_1 related to d . This vertex is not adjacent to the white vertex a_2 . Hence a_2 remains white at the end of turn k . Moreover, by Lemma 2.12, we have $C_k^+(d) \cap D_k = \emptyset$. This implies that Dominator will not play in $C_k^+(d)$ until the end of Phase 1.0. In other words, Dominator will play all his remaining moves in Phase 1.0 in $C_k^-(d)$. But $a_2 \in W_k$ is not adjacent to any vertex in $D_k^-(d)$. Hence, Dominator's further moves in Phase 1.0 will not change the color of a_2 .

We now prove (ii). Assume first that $k' > k$. By (i), the P_5 -child of d' belongs to $W_{k'}$. But d has turned blue at turn k . Hence d cannot be the P_5 -child of d' . Suppose now that $k' < k$. By Lemma 2.12, $C_{k-1}^+(d') \cap D_{k-1}$ is empty. Therefore the P_5 -child of d' does not belong to D_{k-1} . But to be highlighted at turn k , d has to belong to D_{k-1} . We conclude as previously. \square

We are now ready to state the following proposition, which will directly imply our main result for D-game.

Proposition 2.14 *Let F be a weakly $S(K_{1,3})$ -free forest. For all $k \in \{0, \dots, k^{**}\}$, define*

$$S_k = \sum_{i=1}^k e_i \text{ and } K_k = \sum_{v \in B_k} w_k(v). \text{ The following statements are true.}$$

- $I_k : K_k \leq S_k$.
- $II_k : \forall v \in B_k \setminus H_k$, if $w_k(v) = 0$, then $C_k^-(v) = \emptyset$ and $C_k^+(v) \cap D_k = \emptyset$.
- $III_k : \forall v \in B_k$, if $w_k(v) = 0$, then v is not a critical center at the end of turn k .

Proof. We will proceed by induction on k . Since B_0 is empty and $K_0 = S_0 = 0$, the three statements are trivially true for $k = 0$. Suppose that I_k , II_k and III_k are true for some $k \in \{0, \dots, k^{**} - 1\}$. We denote by b_{k+1} the number of vertices which turn blue during turn $k + 1$.

First, we deal with the weighting process for Dominator's turn. We assume that turn $k + 1$ belongs to Dominator. At least two vertices turn from white to red and Dominator earns 6 points with these two vertices. Hence, $e_{k+1} \geq b_{k+1} - 1$, which implies that $S_{k+1} \geq S_k + b_{k+1} - 1$. During this turn the weight of the vertices in B_k does not change and only $b_{k+1} - 1$ new blue vertices get weight 1 (the highlighted vertex gets weight 0). Therefore, $K_{k+1} \leq K_k + b_{k+1} - 1$. Using the induction hypothesis, it yields $K_{k+1} \leq S_{k+1}$. We conclude that I_{k+1} is true.

Let w be a vertex in B_k , such that $w_k(w) = 0$. Since $C_{k+1}^+(w) \cap D_{k+1} \subseteq C_k^+(w) \cap D_k$ and $C_{k+1}^-(w) \subseteq C_k^-(w)$, the truth of statement II_{k+1} is straightforward from the induction hypothesis II_k for the old blue vertices whose weight remains 0. In other words, we only have to pay attention to blue vertices, new or old, which get new weight 0 during turn $k + 1$. Here, the weight of the former blue vertices does not change during this turn and all the new blue vertices which are not in H_{k+1} get weight 1. Hence II_{k+1} is trivially true. By Observation 2.3, in order to prove III_{k+1} , we only need to pay attention to vertices whose weight turns to 0 during this turn. Here, it is the case for an unique vertex, the dangerous center $d \in D_k$, which is elected by Dominator and highlighted during turn $k + 1$. We emphasize that this is the only point of the whole proof where the weakly $S(K_{1,3})$ -free condition is used. Since Dominator follows the breaking- P_5 strategy, it implies that the vertex a_1 (defined in Lemma 2.8) is played by Dominator at turn $k + 1$. Hence, a_1 belongs to R_{k+1} . Let now P be a path in $\mathcal{P}(F)$ whose center is d . By Lemma 2.8, P goes through a_1 which is in R_k . Therefore P cannot be a critical P_5 in the residual forest F_{k+1} , which in turn implies that d is not a critical center at turn $k + 1$. In conclusion III_{k+1} holds.

Now, we prove the heredity of our statements for the weighting process of Staller's turn. We suppose that turn $k + 1$ belongs to Staller and denote by v the vertex played by her. We have to deal with the five different cases of the weighting process. We start by proving I_{k+1} . For the two first cases, Staller earns 3 points with the vertex v which turns from white to red. Hence $e_{k+1} \geq b_{k+1}$ and $S_{k+1} \geq S_k + b_{k+1}$.

Case (A). Because all new blue vertices get weight 1 and the weight of former blue vertices does not change, we have $K_{k+1} \leq K_k + b_{k+1}$. By induction hypothesis, $K_k \leq S_k$, so we conclude that $W_{k+1} \leq S_{k+1}$.

Case (B). Since v is the P_5 -child of the vertex $d \in H_k$, it is adjacent to a leaf l of F . Since v belongs to W_k , this leaf is also in W_k . Hence, at least two vertices, v and l turn from white to red during turn $k + 1$. It implies that $e_{k+1} \geq 3$ and $S_{k+1} \geq S_k + 3$. Since all new blue vertices get weight 0 and d is the only former blue vertex whose weight is increased to 1, we get $K_{k+1} \leq K_k + 1 \leq S_k + 1 < S_{k+1}$.

For the three last cases, the vertex v played by Staller turns from blue to red and Staller earns 2 points with this vertex. Hence, $e_{k+1} \geq b_{k+1} - 1$ and $S_{k+1} \geq S_k + b_{k+1} - 1$.

Case (C). If $w_k(v) = 0$, then all new blue vertices get weight 0 and $K_{k+1} \leq K_k$. By induction, we get $K_{k+1} \leq S_k \leq S_{k+1}$. Now, if $w_k(v) = 1$, all new blue vertices get weight 1. But v will not be blue anymore, that is to say $v \notin B_{k+1}$. Hence, its weight 1 will not count for K_{k+1} . Therefore, $K_{k+1} \leq K_k + b_{k+1} - 1 \leq S_{k+1}$.

Case (D). As above, the weight of v will not count for K_{k+1} . Hence $K_{k+1} \leq K_k + 1 - w_k(v)$. Since v is the P_5 -child of a vertex $d \in H_k \subseteq B_k$, $C_k^-(v)$ contains at least d . By Lemma 2.13 (ii), v is not in H_k . Hence, the induction hypothesis II_k yields $w_k(v) = 1$. In conclusion, $K_{k+1} \leq K_k + 1 - w_k(v) \leq K_k \leq S_k \leq S_{k+1}$.

Case (E). Let a_2 be the P_5 -child of v .

- (a) If $a_2 \in W_k$, then $K_{k+1} \leq K_k + b_{k+1} - 1$, because a_2 gets weight 0. By induction, $K_{k+1} \leq S_k + b_{k+1} - 1 \leq S_{k+1}$.
- (b) If $a_2 \in B_k$, a_2 gets new weight 0. Therefore, $K_{k+1} \leq K_k + b_{k+1} - w_k(a_2)$. Since $v \in H_k \subseteq B_k$, $C_k^-(a_2)$ contains v . By Lemma 2.13, a_2 is not in H_k . Finally, II_k implies that $w_k(a_2) = 1$, which in turn implies $K_{k+1} \leq K_k + b_{k+1} - 1 \leq S_{k+1}$.
- (c) If $a_2 \in R_k$, we have $K_{k+1} \leq K_k + b_{k+1} - w_k(v)$, because v does not belong to B_{k+1} . By Lemma 2.13, we know that a_2 has turned red some moves after the moment v has been highlighted. Moreover, it happened during a Staller's turn. There are two possibilities. First, a_2 has turned red because it has been played by Staller at turn $k' < k + 1$. Case B or Case D of the weighting process for Staller's turn has been applied. Hence, during this turn, v has gotten weight 1. Second, a_2 might have turned red without being played by Staller. The additional weighting process ensures that d has also gotten weight 1. Moreover, the only weighting process, which can change the former weight of a blue vertex into 0, is the second item of Case E for Staller's turn. But, this process is applied only if v is the P_5 -child of another highlighted vertex. Once more, Lemma 2.13 ensures that it is impossible because v is itself highlighted. In all the cases $w_k(v) = w_{k'}(v) = 1$ and we can conclude as before.

We now prove II_{k+1} and III_{k+1} . As for Dominator's turn, we only have to pay attention to the blue vertices, whose weight becomes 0 during turn $k + 1$.

Case (A). None of the new blue vertices gets weight 0 and no former blue vertex weight change from 1 to 0. Then, II_{k+1} and III_{k+1} are trivially true.

Case (B). The only vertices whose weight turn to 0 during the turn are the new blue vertices in $N(v)$. Let u be such a vertex. The vertex v is the P_5 -child of a vertex $d \in H_k$. Note that because d was already blue at the beginning of turn $k+1$, we have $d \neq u$. The vertex d is the only neighbor of v whose height is strictly less than the height of v . Hence $h(u) > h(v) > h(d)$. Since v is now red, $C_{k+1}^-(u)$ is empty. The vertex d has been highlighted during a Dominator's turn, say turn $k' < k+1$. Since v and u are in $W_{k'}$, u belongs to $C_{k'}^+(d)$. By Lemma 2.11, $C_{k+1}^+(u) \cap D_{k+1} \subseteq C_{k'}^+(u) \cap D_{k'} \subseteq C_{k'}^+(d) \cap D_{k'}$. By Lemma 2.12, the last set of these inclusion list is empty. Thus $C_{k+1}^+(u) \cap D_{k+1}$ is also empty. In conclusion II_{k+1} is proved. Moreover, we have $C_{k'}^+(d) \cap D_{k'} = \emptyset$ and $u \in C_{k'}^+(d)$. Hence u is not in $D_{k'}$, but it is in $W_{k'}$. By Lemma 2.7, u is not a critical center at the end of turn $k+1$. In conclusion, III_{k+1} holds.

Case (C). If $w_k(v) = 1$, then no blue vertices, old or new, would see their weight turn to 0. In that case II_{k+1} and III_{k+1} are trivially true. Suppose that $w_k(v) = 0$ and let $u \in N(v)$ be a new blue vertex. The vertex v is not in H_k . By induction hypothesis II_k , $C_k^-(v) = \emptyset$ and $C_k^+(v) \cap D_k = \emptyset$. Hence, u has to belong to $C_k^+(v) \setminus D_k$. By Lemma 2.7, u cannot be a critical center at turn $k+1$. That proves III_{k+1} . By Lemma 2.11, $C_{k+1}^+(u) \subseteq C_{k+1}^+(v)$. Therefore $C_{k+1}^+(u) \cap D_{k+1} = \emptyset$. Finally $C_{k+1}^-(u) = \emptyset$, because v is red at the end of turn $k+1$. In conclusion II_{k+1} is true.

Case (D). The proof is the same as for Case (B).

Case (E). Let a_2 be the P_5 -child of v . For subcases (a) and (b), this vertex is the only blue vertex which gets weight 0 during this turn. The vertex a_2 belongs to $C_k^+(v)$. It implies that $C_{k+1}^+(a_2) \subseteq C_k^+(a_2) \subseteq C_k^+(v)$ and $C_{k+1}^+(a_2) \cap D_{k+1} \subseteq C_k^+(v) \cap D_k$. Since v is an highlighted vertex, $C_k^+(v) \cap D_k$ is empty by Lemma 2.12. Hence $C_{k+1}^+(a_2) \cap D_{k+1}$ is also empty. Moreover, because v is red at the end of turn $k+1$, we have $C_{k+1}^-(a_2) = \emptyset$. That shows that II_{k+1} holds. Since $v \in H_k$, this vertex has been highlighted during a Dominator's turn, say turn $k' \leq k$. We have $C_{k'}^+(v) \cap D_{k'} = \emptyset$. Hence, a_2 does not belongs to $D_{k'}$. By Lemma 2.13, a_2 belongs to $W_{k'}$. Therefore, applying Lemma 2.7, the vertex a_2 is not a critical center at the end of turn $k+1$. In conclusion III_{k+1} is true. Finally, for subcase (c), no blue vertex get new weight 0, so there is nothing to prove.

Finally, we have to prove that applying the additional weighting process at the end of turn $k+1$ does not change the true of statement I_{k+1} , II_{k+1} and III_{k+1} . Since no blue vertex gets weight 0 during this process, II_{k+1} and III_{k+1} remain trivially true. Let x_1, \dots, x_m be the P_5 -children of some vertices in H_k , which turn red without being played by one of the player. After applying this process, the new value of K_{k+1} will be less or equal to $K_{k+1} + m - w_k(x_1) - \dots - w_k(x_m)$. By the P_5 -child definition, $C_k^-(x_i) \cap H_k$ is non empty, for any $i \in \{1, \dots, m\}$. Moreover, by Lemma 2.13, we have $x_i \notin H_k$. Applying the induction hypothesis II_k , we get that $w_k(x_i) = 1$. That proves that the value of K_{k+1} is not increased by the application of the additional process. We conclude that I_{k+1} remains true. \square

Theorem 2.15 *If F is a weakly $S(K_{1,3})$ -free forest on n vertices, then $\gamma_g(F) \leq \frac{3n}{5}$.*

Proof. By Proposition 2.9, we only have to prove that $c^{**} \leq e^{**}$. Let $K_{k^{**}}$ and $S_{k^{**}}$ be defined as in Proposition 2.14. Note that $S_{k^{**}} = e^{**}$. Statement $III_{k^{**}}$ from Proposition 2.14 implies that $c^{**} \leq K_{k^{**}}$. Moreover, statement $I_{k^{**}}$ implies that $K_{k^{**}} \leq S_{k^{**}}$. We conclude that $c^{**} \leq e^{**}$. \square

2.3 The proof for S-game

The proof for S-game will proceed in the same way as for D-game, except that now we introduce Phase 0 of the game, which corresponds to the first move of Staller. Hence, Phase 1.1 will now start at turn $k = 2$. We still write k^{**} for the last turn of Phase 1.1 and c^{**} for the number of critical centers at the end of turn k^{**} . We set $e_1 = p(F_0) - p(F_1) - 5$. For $k \geq 2$, e_i is defined as for D-game. We also set $e^{**} = e_2 + \dots + e_{k^{**}}$.

In the proof of [9, Theorem 2], it is proved that, by following the greedy strategy in S-game for an isolate-free forest on n vertices, Dominator can force the game to end in at most $\frac{3n - e_1 - e^{**} + c^{**}}{5}$ turns. Here, e^* is the amount of bonus point earned during Phase 1 and c^* is the number of critical centers at the end of this same phase. As for D-game, this statement implies the following proposition.

Proposition 2.16 *Let F be an isolate-free forest on n vertices. If Dominator plays S-game according to the breaking- P_5 strategy, then the number of turns will be at most $\frac{3n - e_1 - e^{**} + c^{**}}{5}$.*

We can define the weighting and the highlighting process exactly in the same manner as for D-game. We can state a proposition very similar to Proposition 2.14. Only statement I_k will change. For this new proposition, we will have $K_k \leq S_k + 2$, for all $k \geq 0$. Indeed e_1 is equal to the number of blue vertices minus 2. Since all these blue vertices will get weight 1, we have $K_1 \leq S_1 + 2$. Finally, this new proposition will directly prove that $c^{**} \leq e_1 + e^{**} + 2$. It clearly implies the below theorem.

Theorem 2.17 *If F is a weakly $S(K_{1,3})$ -free forest on n vertices, then $\gamma'_g(F) \leq \frac{3n + 2}{5}$.*

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